Virtual Logic—Number and Imagination

Louis H. Kauffman

I. Introduction

This column consists in a dialogue (sections 2 and 3) between Cookie and Parabel, protagonists from an earlier column called “Fragments of the Void” (Kauffman, 2004). The dialogue is about the nature of number and the emergence of form and number from the void. In this discussion the void is not empty. It is quite full. The void consists in what is given prior to any distinction. Thus in the void there is no distinction between the observer and the observed or between concept and reality. There are no distinctions in the void, and all that we are familiar with and all that we are not familiar with is right there, but there is no there there. This property of everything and nothing (but the void does not even have this property) makes the void maximally unstable, ready to exfoliate form (we take the form of distinction for the form) at an instant if there were such a spark as an instant.

Cookie and Parabel show how number arises from the void and how it is our imagination, our reflexivity that produces and maintains the apparent infinity of entities. In taking this point of view on cognition and form we see that mathematics and indeed the arising of all forms and worlds is reflexive and must include the observer, for it is only with the imagination of an observer that these worlds come to life. In the absence of an observer, the apparent worlds fall back into the potentiality of the void.

At the end of the column a reference list forms a background to the dialogue. These include previous papers of the author (Kauffman, 1987, 2004, 2005, 2009, 2010); Laws of Form by G. Spencer-Brown (1969); a paper on Number by Warren McCulloch (1965); a warning by the mathematician Edward Nelson (2006); and the Tractatus Logicus Philosophicus by Ludwig Wittgenstein (1922).

II. From the Void to Infinity — A Dialogue

In this dialogue, Cookie and Parabel have begun talking about the nature of numbers. They are concerned with the well-known philosophical question:

Why is there something rather than nothing?

From their point of view, a place to start would be how numbers arise and how it comes to pass that there are infinitely many numbers in our apparently finite world.

1. Louis H. Kauffman is professor for mathematics at the Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607-7045. E-mail: kauffman@uic.edu
**Cookie:** Counting numbers are a good place to start. We can represent the numbers by collections of sticks, or in typography, vertical slashes:

1 = |
2 = ||
3 = |||
4 = ||||
5 = |||||

with the number $n$ having the form

$$n = |||...||$$

where there are exactly $n$ vertical slashes representing that number.

**Parabel:** Well Cookie, that looks simple enough, but don't you need to know what a number is already in order to say what you just said?

**Cookie:** Oh dear Parabel, you know that is the case. I cannot speak to you about number unless you already have the concept of number, and the concept of a multiplicity of things. All I can do is show you how to work in a symbol system for numbers. For example, in my system above we can add so easily. I just take the slashes for $n$ and place them next to the slashes for $m$. For example

$$2 + 2 = || || = |||| = 4.$$

In general $n + m = nm$ where $nm$ denotes the row of slashes for $n$ followed by the row of slashes for $m$. So this system lets me define arithmetic and to go into the deep waters of number theory.

**Parabel:** You want to lead me into an obsession and make my philosophical parts go right to sleep. No, Cookie, you do not fool me. I want to know where those numbers come from. I want to see the source of number from that which is not number.

**Cookie:** I can only try. Consider the following diagram. I begin with a line and I let it begin to turn on itself.
Then I let this process of self reference continue.
And finally the line manages to slide free of itself, and a curl and an anti-curl have appeared!

I have shown how an undifferentiated space (the original line) can turn upon itself, go through self-reference, and emerge with the production of two distinct polar entities, the analogs of +1 and -1, polar opposites that can annihilate each other in a return to the void of the original line. (Just read the sequence of events backwards.). Once we show how distinct entities can emerge from a void, we are on the track toward number and multiplicity.

**Parabel:** I could do that manipulation with a bit of string and a flat table on which to move the string. But you raise curious issues here. You suggest that the undifferentiated void, the inchoate place that just is, before any distinctions occur, can give rise to that which is distinct and indeed to polar opposites, particles that can annihilate each other and return to void.

**Cookie:** Indeed!

**Parabel:** In your model, the “void” was a string, but it moved and was differentiated from a background that was the plane of the drawing or a flat space for the string. So your void was a fake. And there already was a distinction. You showed how polar opposites would emerge in the relation of one void to another.

**Cookie:** It was not intended to be a swindle. There is no way to illustrate or present the emergence of form from void by really using the void. There is no way to “use” the void because, well, it is not tangible. The void has no properties whatsoever and it is not available as a backdrop. Thus the best I can do is hold a bit of theatre as I did with this string. The table or flat space is the stage of the theatre. The string is the actor. The string is the void, and with the help of the backdrop we can imagine the void moving upon itself and giving birth to opposites. Now you have to take your imagination and let go of the backdrop. Imagine the void undergoing this process but
undergoing it intrinsically within itself and giving birth. The void ignites into polar opposites!

**Parabel:** I begin to see your point. But how can something with no attributes, no distinctions, nothing, give rise to something? Where do these polar opposites come from? And how do we come to know them?

**Cookie:** Consider that void carefully. It is devoid of all characteristics, and so it is even devoid of being devoid of all characteristics! The void therefore DOES have characteristics for if it did not have ANY characteristics that too would be a characteristic. The void is not what you think it is. The void is not a blank. It is very full. And it is full exactly of opposites! Think of anything at all. Well the void is not that and so must have the opposite characteristic. Is the void cold? No. Then it is hot. No. Then it is hot and cold. That is closer but if you try to pin that down you find that the void is hot, the void is cold, the void is not hot, the void is not cold. The void is neither hot nor is it cold. The void exists, the void does not exist, the void neither exists nor does it not exist. The void is all. The void is nothing. The void is neither all nor nothing. I hope you see that the void is the wonderful generator of opposites. Of course it is not really a generator of opposites. The void generates nothing and everything.

**Parabel:** Whew. You are very persuasive, but there is not the slightest content in what you say. Polar opposites emerged from your void. And the model illustrates something that all your talk does not show. The polar curls that emerge from your void were not “in” the void before the void turned on itself and produced them. So the void was and was not composed of these opposites. You may try to speak paradoxically, but your model is more sensible than that. It can differentiate between actuality and potentiality.

**Cookie:** Stop right there. It is not the void that differentiates between actuality and potentiality. These are our concepts. In the void there is no difference between actuality and potentiality. All forms are within the void. No forms are in the void.

**Parabel:** Now you are infuriating me. I think that the void is just a word the way you use it. It is nothing but a placeholder for your talk about everything and nothing!

**Cookie:** Not at all. What I call the void is what we have before conceptualization. And you will have to admit that conceptualized entities are fragile and subject to language. I am positing a rich and varied and yet undifferentiated world that lies outside of your concepts and outside of my concepts. But this “world” is always at work with us. If it were not for this undifferentiated world, I should have nothing to discriminate. I would not have the morning sunlight or the evening star. There really would be nothing at all. Ludwig Wittgenstein (1922) said it when he said “Whereof we cannot
speak, we must remain silent” (p. 151) and he also said “Feeling of the world as a limited whole—it is this that is the mystical” (p. 149), and he said “The limits of my language are the limits of my world” (p. 115). The world of which he speaks is the fragile world of concepts, of pictures of reality in language. He knew that there is much more than this. There is a creative source and this creative source I have chosen to call the void.

**Parabel:** Then what is the direct nature of this creative source? What power lies behind it? What is its mechanism?

**Cookie:** Mechanism, power? You are projecting your favorite concepts to the void, but of course the void has mechanism and it has magic. The void has power and it has weakness. It has any of these and none of these. Don't you see? The void is the place where all opposites meet and annihilate one another. The void is the place where all opposites are born.

**Parabel:** Now I am beginning to think that your void is like a black hole and its ability to produce entities is the same as the story the physicists tell us about the Hawking Radiation!

**Cookie:** Perhaps you should modify your text so that our readers can follow this part of the conversation?

**Parabel:** Black holes are places in the universe where the gravitational field is so strong that even light cannot get out. If anything falls into a black hole it will, putatively, never be seen again. But Stephen Hawking made a telling observation about the structure of black holes. He pointed out that if a photon should split into an electron and a positron (its antiparticle) near a black hole, then the positron (say) could be grabbed by the gravity of the black hole and pulled in, while the electron went scooting off!

The result is that the black hole has effectively radiated an electron. If you work it out, the result is that there is a definite probability for a black hole to evaporate and disappear due to the Hawking radiation. It seemed to me that your void was rather like the black hole. It could allow opposites to appear and let one or both of them radiate away from itself.

**Cookie:** I see. But my void is much simpler than Hawking’s black hole. My void is just the propensity of mind to hold opposites. Indeed it is not possible to think A without having not A as well. And in the powerful neutral void there is no preference for A or not A and yet, since all opposites exist in the void, there IS a preference for one of A or not A. In this way distinctions can emerge from the void! But yes, this emergence of distinctions from the void is rather like the Hawking radiation. Maybe the physicists should study our void.
Parabel: Well that took a long time just to tell me that only the void is sensitive enough to produce something from nothing.

Cookie: That does sum it up.

Parabel: Can we go back to numbers now?

Cookie: I was afraid you would never ask.

Parabel: Do you really spend a lot of time thinking about these rows of vertical slashes?

Cookie: I do. But I abbreviate them. I like to use the binary system. You know how that works. We have

\[
\begin{align*}
1 & \rightarrow 1 \\
10 & \rightarrow 2 \\
100 & \rightarrow 4 \\
1000 & \rightarrow 8 \\
10000 & \rightarrow 16 \\
100000 & \rightarrow 32 \\
\end{align*}
\]

and generally 1 with n zeros after it, represents \(2^n\). So these numbers grow astronomically fast and we can represent numbers that we would never write as sequences of vertical slashes. For example what number does this binary number represent?

\[
11111111111111111111111
\]

Parabel: Well, this number has 23 digits and so represents

\[
1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \ldots + 2^{22}
\]

and I recall from my math courses that that is equal to \(2^{23} - 1\), a pretty big number.

Cookie: You are certainly right about that. You could never write the binary number 11111111111111111111111 out as sequence of \(2^{23} - 1\) vertical slashes. And yet we are able to work with this number and calculate with it. By using notational systems for our numbers we are able to extend our reach and work with multiplicities that go far beyond our ability to directly construct them.

Parabel: We are calculating with imaginary quantities in the literal sense of the word imaginary!
Cookie: Yes indeed. Of course, when we do algebra we also obtain facts about numbers that do not depend on their size. For example

\[(a + b)^2 = a^2 + 2ab + b^2\]

is true for any two numbers a and b. But I think we still should worry about the numerals that are represented by sequences of vertical slashes.

|,||,|||,||||,|||||,||||||,||||||||,|||||||||,||||||||||,|||||||||||||,...

How far does it *actually* go? I know that you can say to me, well there is no largest one because if you say that

\[M = \overbrace{|||\ldots|||}^{\text{large number}}\]

is the largest one, then I will say to you, why just add one more slash to M and it will be larger.

Parabel: Yes. And I will say to you. No, no, no!! Adding one more slash to the largest one will not make it bigger. We no longer have the ability to compare a truly large M and M+1 (M with one more slash). Our ability to count and compare has succumbed to the challenge of the size of such large numbers. As far as we are concerned M = M+1. This also means that if you take a slash away from M it also does not change size. So you will not be able to whittle down M by taking slashes away from it.

Cookie: It is devilish to consider a number like large M. An M like this is almost the same as our previous ideas about infinity. But even with infinity we usually assume that we can operate on all the parts and match them up. For example we can match each number n to a corresponding even number 2n, showing that the infinity of all positive integer numbers and the infinity of the even numbers have the same size. But with a largest finite number M, there is no way that we can make this argument.

Parabel: But then how do we know that M is finite? If it is too big to ever determine whether it is finite, then I don't see the difference between M and infinity! It seems that it is like mathematical infinity, but we know even less about M than we do about infinity.

Cookie: You are right about that but M is more like a practical version of infinity. I can make models of such systems by using the usual number system as the “outside world” O and saying that the persons who cannot measure M are living in a more restricted world W. The inhabitants of W might for example be definitely unable to deal with \[2^{2^{2^{2^{2}}}}\] and we could take M equal to that as far as the outside world is concerned. We in the outside world could point to their activities and make comments, but some of our comments would be meaningless to them. For example,
we might see them saying things like \( M + 1 = M \) and changing their value of \( M \) in the process, but we would disagree and say that \( M+1 \) is not equal to \( M \). Or (depending on what the people in \( W \) do) we might say, you have an \( M \), but you keep changing the value of it!

**Parabel:** It is very singular to me that the \( W \)-people could (just like us) speak about numbers that are larger than \( M \). Do they KNOW that they can do this. I suggest that \( M \) should be so large that it is beyond their capacity to notate any numbers that are larger than \( M \). Is this a consistent position?

**Cookie:** You are zeroing (pardon the expression) in on a key question. Lets take it very carefully. We can notate very large numbers. We can notate numbers way beyond our capacity to make numerals for them (a numeral is a number in the form of a row of vertical slashes). It would seem that there is no limit to the possibilities of our notation, but this is not so. For every type of notation that I might invent there is always the possibility of repeating this notation directly a specific number of times.

This consideration can be iterated of course, but there is a limit to the reaches of notation. It would be hard to estimate it, but there are numbers that are just so large that we cannot achieve inductive definitions that will reach them.

**Parabel:** So you are telling me that there are inaccessible numbers, numbers that we will never, due to our temporal and physical limitations, devise notations for them much less numerals for them.

**Cookie:** Indeed that is what I am saying. Thus if we wish to speak of “largest numbers” we shall have to speak of numbers \( M \) that are inaccessible in this way. Such numbers are imagined by us, but we will never be able to touch them with notation. Therefore we must ask, do such numbers exist? They are imaginary, but then so are all mathematical “objects.”

**Parabel:** In all formal systems that use an infinite number of words the actual elements of the formal system are imaginary. That is, there is no way to actually write all words that are in the system, so we are in the air.

In arithmetic we use ||,|||,||||,|||||,... and any formal number theory must do this, but we only imagine the idea of writing all of the |||...|||. If fact, we can only write finitely many of them. We refer to the imaginary values of the unwritable symbols in the formal system. We are not referring to numerals when we prove anything about a formal system of this type. We have to go beyond numerals to imagined quantities.

This collapses the notion that mathematics is a game played with symbols on a piece of paper. The formal system of symbols on the piece of paper wraps around into the cognition and imagination of the mathematician. The observer is part of the picture. Without that there is no mathematics.
A necessary reflexivity lies beneath any attempt to do mathematics. Once this is understood, then completed infinities, like the natural numbers, are welcome as exterior descriptions of the properties of the imaginary values that we necessarily use and explore.

**Cookie:** And so we can make entities like $W = |||||\ldots$ this is an infinite row of vertical slashes, an idealization of the very large.

But $W$ becomes an object for us with the simple property that

$$W = |W.$$  

$W$ sits inside its own indicational space. $W$ is a mathematical image of infinity and at the same time a mathematical image of the reflexive wrapping of imagination and apparent reality.

### III. Epilogue

**Parabel:** Well, do you think we managed to get somewhere in that conversation?

**Cookie:** Yes Parabel, I think we did make some progress. We saw that mathematics and number are invented in the course of the cognition that arises from the void, from the black box of everything/nothing that is the potential well of every conversation. And we saw that infinity arises in a formal way from the attempt to see all the numbers, but that infinity is not really all numbers. Infinity, even when completed, is just a formal representation of the possibility of always going on.

**Parabel:** Yes. And Infinity represents the circularity of our involvement in the process of extracting and inventing thoughts and formalisms in this unlimited way. Infinity and reflexivity are in the form identical.

### References


McCulloch, W. S. (1965). What is a number that a man may know it, and a man, that he may know a number? In Embodiments of mind (pp. 1-18). Cambridge, MA: The MIT Press.

